Solutions

Exam 2 Practice Problems
Chapter 3 and 4.1-4.5

---

1. Find the first and second derivative

- $x^3 - 3(x^2 + \pi^2)$
- $(x+1)^2 e^{x^3}$
- $\ln(\cos(1/x)))$
- $\cot^3(2/t)$
- $(2x+1)\sqrt{2x+1}$
- 9^{2t}
- $\frac{\sqrt{t}}{1+\sqrt{t}}$
- $\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$

1st	2nd
$3x^2 - 6x$	$6x - 6$
$(x+1)^2 \cdot e^{x^3} \cdot 3x^2 + 2(x+1)e^{x^3}$ $=$ $e^{x^3}(x+1)[3x^3 + 3x^2 + 2]$ $=$ $e^{x^3}[3x^4 + 6x^3 + 3x^2 + 2x + 2]$	$e^{x^3}[12x^3 + 18x^2 + 6x + 2]$ $+ e^{x^3} \cdot 3x^2 [3x^4 + 6x^3 + 3x^2 + 2x + 2]$
$\frac{1}{\cos(1/x)} \cdot \sin(1/x) \cdot \frac{-1}{x^2}$ $=$ $-\frac{\cancel{\ }\tan(1/x)}{x^2}$	$-\frac{x^2 \cdot \sec^2(1/x) \cdot \frac{-1}{x^2} - \tan(1/x) \cdot 2x}{x^4}$
$3\cot^2(2/t) \cdot (-\csc^2(2/t)) \cdot \left(-\frac{2}{t^2}\right)$ $=$ $6\left(\frac{\cot(2/t) \cdot \csc(2/t)}{t}\right)^2$	$12\left(\frac{\cot(2/t)\csc(2/t)}{t}\right) \times$ $\cancel{t(\cot(2/t) \cdot -\csc(2/t) \cdot \cot(2/t)}$ $\times \frac{t(\cot(2/t) \cdot -\csc(2/t)\cot(2/t) \cdot \frac{-2}{t^2}) - \cot(2/t)\csc(2/t)}{t^2}$
Rewrite $(2x+1)\sqrt{2x+1}$ $=$ $(2x+1)^{3/2}$ $\Rightarrow$ $\frac{3}{2}(2x+1)^{1/2} \cdot 2 = 3\sqrt{2x+1}$	$\frac{3}{2}(2x+1)^{-1/2} \cdot 2 = \frac{3}{\sqrt{2x+1}}$

1st	2nd
$2\ln(9)\cdot 9^{2t}$	$(2\ln(9))^2\cdot 9^{2t}$
$\dfrac{(1+\sqrt{t})\cdot\frac{1}{2\sqrt{t}}-\sqrt{t}\cdot\frac{1}{2\sqrt{t}}}{(1+\sqrt{t})^2}=\dfrac{1}{2\sqrt{t}(1+\sqrt{t})^2}$	$\dfrac{2\sqrt{t}(1+\sqrt{t})^2\cdot 0-(1(2\cdot\frac{1}{2\sqrt{t}}(1+\sqrt{t})^2+2\sqrt{t}\cdot 2(1+\sqrt{t})}{4t(1+\sqrt{t})^4}$ $\parallel$ $-\dfrac{\frac{(1+\sqrt{t})}{\sqrt{t}}((1+\sqrt{t})+2\sqrt{t})}{4t(1+\sqrt{t})^4}=\dfrac{-1-3\sqrt{t}}{4t^{3/2}(1+\sqrt{t})}$
$\frac{1}{4}(e^{4x}+4xe^{4x})-\frac{1}{16}\cdot 4e^{4x}$ $\parallel$ xe^{4x}	$e^{4x}+4xe^{4x}=e^{4x}(1+4x)$

2. **Use implicit differentiation to find dy/dx and dx/dy**

- $x^2y^2 = 1$
- $y^2 = \sqrt{\frac{1+x}{1-x}}$
- $x^y = \sqrt{2}$
- $ye^{\tan^{-1}x} = 2$
- $5x^{4/5} = 10y^{6/5}$

dy/dx	dx/dy
$x^2 \cdot 2y \cdot y' + 2x \cdot y^2 = 0$ $y' = -\frac{y}{x}$	$x^2 \cdot 2y + 2x \cdot x' \cdot y^2 = 0$ $x' = -\frac{x}{y}$
$2y \cdot y' = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2} \cdot \frac{(1-x)\cdot 1 - (1+x)\cdot(-1)}{(1-x)^2}$ $= \frac{1}{\sqrt{(1+x)(1-x)^3}}$ $y' = \frac{1}{2y\sqrt{(1+x)(1-x)^3}}$	$2y = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2} \cdot \frac{(1-x)\cdot x' - (1-x)(-x')}{(1-x)^2}$ $= \frac{x'}{\sqrt{(1+x)(1-x)^3}}$ $x' = 2y\sqrt{(1+x)(1-x)^3}$
Rewrite $x^y = \sqrt{2}$ $\Updownarrow$ $\ln x \cdot y = \sqrt{2}$ $(\ln x \cdot y' + \frac{y}{x})x^y = 0$ $y' = -\frac{y}{x \ln x}$	$(\ln x + \frac{y}{x} \cdot x')x^y = 0$ $x' = -\ln x \cdot \frac{x}{y}$
$y' \cdot e^{\tan^{-1}x} + ye^{\tan^{-1}x} \cdot \frac{1}{1+x^2} = 0$ $y' = -\frac{y}{1+x^2}$	$e^{\tan^{-1}x} + ye^{\tan^{-1}x} \cdot \frac{1}{1+x^2} \cdot x' = 0$ $x' = -\frac{1+x^2}{y}$

dy/dx	dx/dy
$4x^{-1/5} = 12y^{1/5} \cdot y'$	$4x^{-1/5} \cdot x' = 12y^{1/5}$
$y' = \frac{1}{3(xy)^{1/5}}$	$x' = 3(xy)^{1/5}$

3. Word Problems (Interpreting the Derivative and Related Rates)

- The surface area S of a right, circular cylinder is related to the base radius and height by the formula
$$S = 2\pi r^2 + 2\pi rh.$$
Assuming height is constant, how is dS/dt related to dr/dt? (For a more difficult problem, do not assume anything is constant and relate dS/dt to both dr/dt and dh/dt).

- Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).

- The volume of a cube is increasing at a rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instance?

- The position at time $t \geq 0$ of a particle moving along a coordinate line is
$$s = 10\cos(t + \pi/4).$$
Find the particle's starting position, furthest distance left and right, and its velocity,speed and acceleration.

- Find the values of h, k and a that make the circle $(x-h)^2 + (y-k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1,2) and that also makes the second derivative of the two curves equal at this point.

- A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = (3 - \frac{x}{40})^2$. Write an expression for the total revenue per trip received by the bus company and find the maximum revenue.

More difficult: $\frac{dS}{dt} = 4\pi r\cdot\frac{dr}{dt} + 2\pi\left(r\cdot\frac{dh}{dt} + \frac{dr}{dt}\cdot h\right) = 2\pi\left(2r\frac{dr}{dt} + h\frac{dr}{dt} + r\frac{dh}{dt}\right)$

If h is constant, then $\frac{dh}{dt} = 0$.

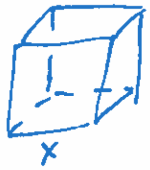
$y' = 3x^2$. Thus the tangent line ~~ta~~ at (a, a^3) is $y = 3a^2(x-a) + a^3$
$= 3a^2x - 2a^3$

If Tangent line intersects curve, then $x^3 = 3a^2x - 2a^3$.

So $0 = x^3 - 3a^2x + 2a^3 = (x-a)(x^2 + ax - 2a^2)$
$= (x-a)(x-a)(x+2a)$

So $x = a$ or $-2a$.

We already knew tangent line intersects at $x=a$. So slope at $x=-2a$ is $3(-2a)^2 = 4\cdot 3a^2$.



$V = x^3$ $\qquad \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$. When $x = 20$cm,

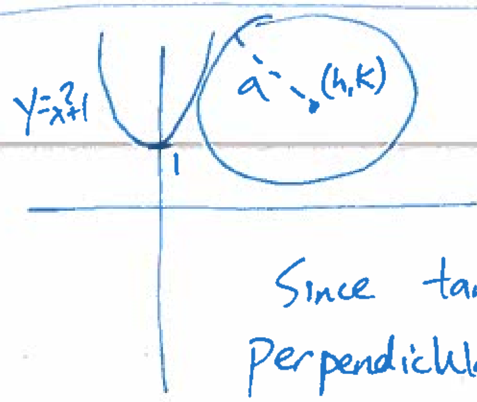
$1200 = 3 \cdot 20^2 \cdot \frac{dx}{dt}$ so that $\frac{dx}{dt} = 1$ cm/min.

$s = 10\cos(t + \pi/4)$. Starting position ($t=0$) is $10\cos(\pi/4) = 5\sqrt{2}$

Max of $\cos(t+\pi/4) = 1$ $\Rightarrow$ Furthest right $= 10$
Min of $\cos(t+\pi/4) = -1$ $\Rightarrow$ Furthest left $= -10$.

$v(t) = s' = -10\sin(t+\pi/4)$, $a(t) = v'(t) = -10\cos(t+\pi/4)$.

speed$(t) = |v(t)| = 10|\sin(t+\pi/4)|$.



$y' = 2x$
$y'' = 2$ $\Rightarrow$ Tangent at $(1,2)$ is $y = 2(x-1)+2 = 2x$.

Since tangent to a circle is perpendicular to radial line, perpendicular line to $y=2x$ at $(1,2)$ is given by

$$y = -\frac{1}{2}(x-1)+2 = -\frac{1}{2}x + \frac{5}{2}.$$

So $\quad k = -\frac{1}{2}h + \frac{5}{2}$ and $a = \sqrt{(2-k)^2 + (1-h)^2}$.

To satisfy 2nd der. requirement, $2(x-h) + 2(y-k)\cdot y' = 0 \Rightarrow y' = -\frac{x-h}{y-k}$.

and $y'' = -\frac{(y-k)\cdot 1 - (x-h)\cdot y'}{(y-k)^2}$ at $(1,2)$ (w/ $y'=2$)

$= \frac{k-2h}{(2-k)^2}$ must equal 2

So combining eqns we get

$h = -4$, $k = \frac{9}{2}$ and $a = \frac{5\sqrt{5}}{2}$.

$0 \le x \le 60$. revenue = price · quantity.

So $r(x) = (3 - \frac{x}{40})^2 \cdot x$.

$r'(x) = (3-\frac{x}{40})^2 + x \cdot 2(3-\frac{x}{40}) \cdot -\frac{1}{40}$

$= 9 - \frac{3}{10}x + \frac{3x^2}{40}$

No critical points, so $x = 60$ corresponds to max of \$135.

4. **Graph Sketching and Extreme Values (Find all critical points, inflection points, relative extrema and absolute extrema. Finish with a sketch of the graph.)**

- $g(x) = \frac{x^2}{4-x^2}$, $\quad -2 < x \leq 1$
- $f(x) = e^{2/x}$
- $h(x) = (2-x^2)^{3/2}$
- $m(x) = \sin x \cos x, \quad 0 \leq x \leq \pi$

$g(x) = \frac{x^2}{4-x^2}$, ~~[illegible]~~ undefined at $x = \pm 2$.

$$g'(x) = \frac{(4-x^2)\cdot 2x - x^2\cdot(-2x)}{(4-x^2)^2} = \frac{8x}{(4-x^2)^2}$$

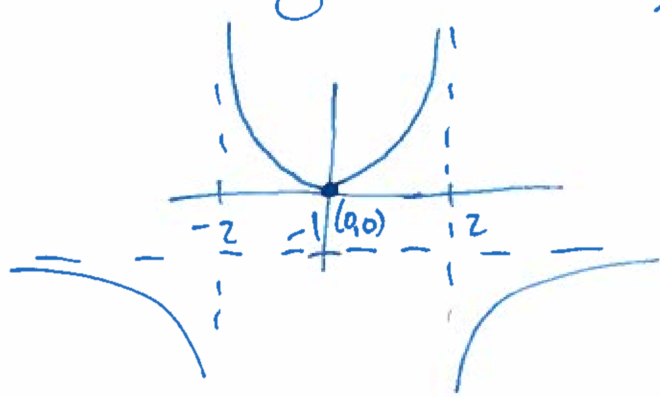
$$g''(x) = \frac{(4-x^2)^2\cdot 8 - 8x\cdot 2(4-x^2)\cdot(-2x)}{(4-x^2)^4} = \frac{8(3x^2+4)}{(4-x^2)^3}$$

crit. pt at $x=0$.

No inflection pts.

$g''(0) > 0$ so min.

		-2		0		2	
g							
g'	$- \; -$		$- \; -$		$+ \; +$		$+ \; +$
g''	$- \; -$		$+ \; +$		$+ \; +$		$- \; -$

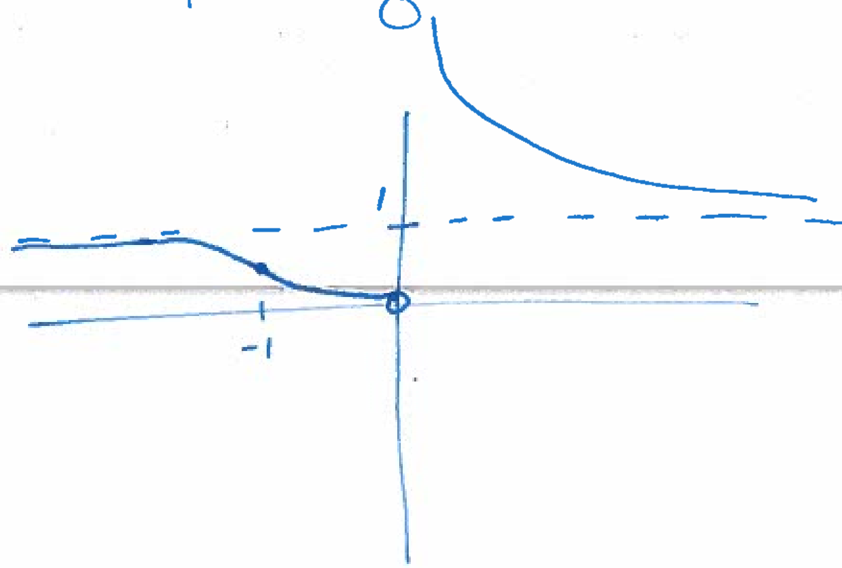


$f(x) = e^{2/x}$ undefined at $x = 0$. Hor. asymp. at $y = 1$

$f'(x) = e^{2/x} \cdot -\frac{2}{x^2} = -\frac{2e^{2/x}}{x^2}$. No crit. pts

$f''(x) = -2\left[\frac{x^2 \cdot e^{2/x} \cdot -\frac{2}{x^2} - e^{2/x} \cdot 2x}{x^4}\right] = +4e^{2/x}\left(\frac{1+x}{x^4}\right)$ possible inflection pt at $x=1$

	$x < -1$	$-1 < x < 0$	$x > 0$
f			
f'	$-$ $-$	$-$ $-$	$-$ $-$
f''	$-$ $-$ $-$	$+$ $+$	$+$ $+$



$h(x) = (2-x^2)^{3/2}$ See Quiz 6.

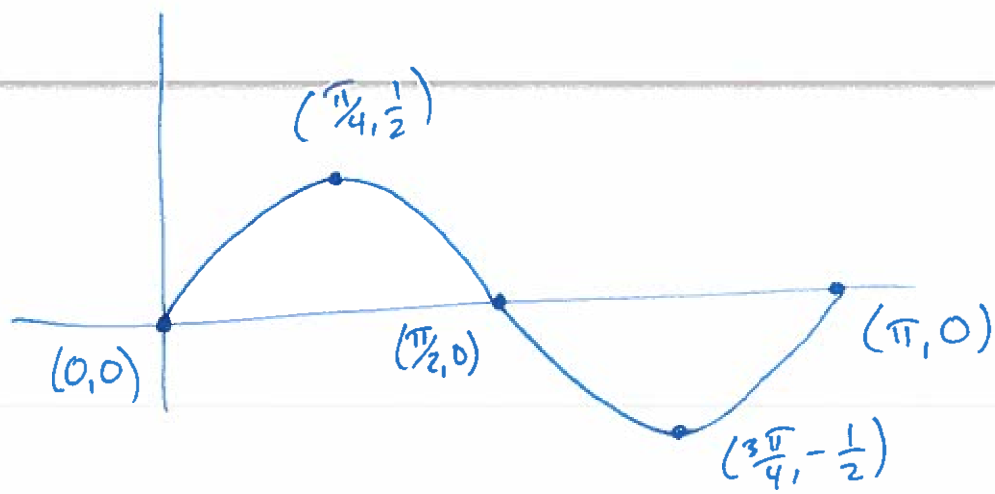
$m(x) = \sin x \cos x \quad , \quad 0 \le x \le \pi$

$m'(x) = -\sin^2 x + \cos^2 x \qquad$ crit. pt. at $x = \frac{\pi}{4}, \frac{3\pi}{4}$

$m''(x) = -2\sin x \cos x + 2\cos x \cdot (-\sin x) = -4 \sin x \cos x$

possible inflec. pts at $x = \frac{\pi}{2}$

	$[0, \frac{\pi}{4}]$	$[\frac{\pi}{4}, \frac{\pi}{2}]$	$[\frac{\pi}{2}, \frac{3\pi}{4}]$	$[\frac{3\pi}{4}, \pi]$
m				
m'	+ +	− −	− −	+ +
m''	− −	− −	+ +	+ +



5. Sketch the general shape of the graph of $y = f(x)$ given y'

- $y' = 2 + x - x^2$
- $y' = x(x-3)^2$
- $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$
- $y' = (x^2 - 2x)(x-5)^2$

$y' = 2 + x - x^2 = -(x^2 - x - 2) = -(x-2)(x+1)$

crit. pts at $x = 2, -1$

$y'' = 1 - 2x$ possible inflect. pt at $x = \frac{1}{2}$.

		-1		$\frac{1}{2}$		2	
y'	− −		+ +		+ +		− −
y''	+ +		+ +		− −		− −

y

$y' = x(x-3)^2$ crit. pts at $x = 0, 3$

$= x^3 - 6x^2 + 9x$

$y'' = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$

possible inflect. pts at $x = 1, 3$

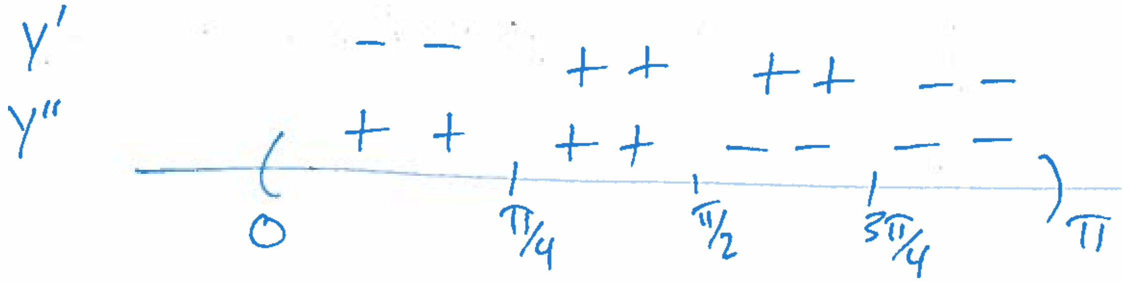
		0		1		3	
y'	− −		+ +		+ +		+ +
y''	+ +		+ +		− −		+ +

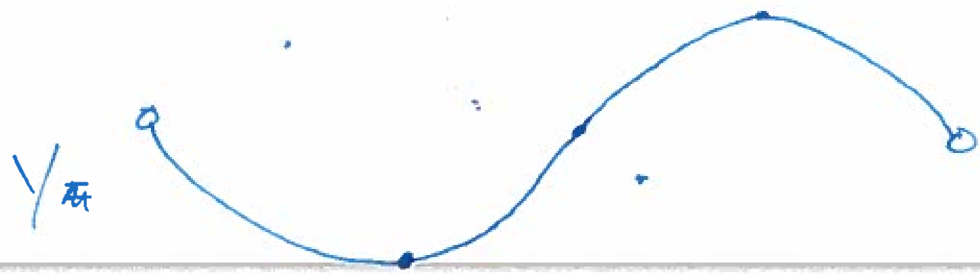
y

$y' = 1 - \cot^2\theta$, $0 < \theta < \pi$

Crit. pt at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$y'' = 2\cot\theta\cdot(-\csc^2\theta) = 2\cot\theta\csc^2\theta$ possible inflection pts at $\theta = \frac{\pi}{2}$



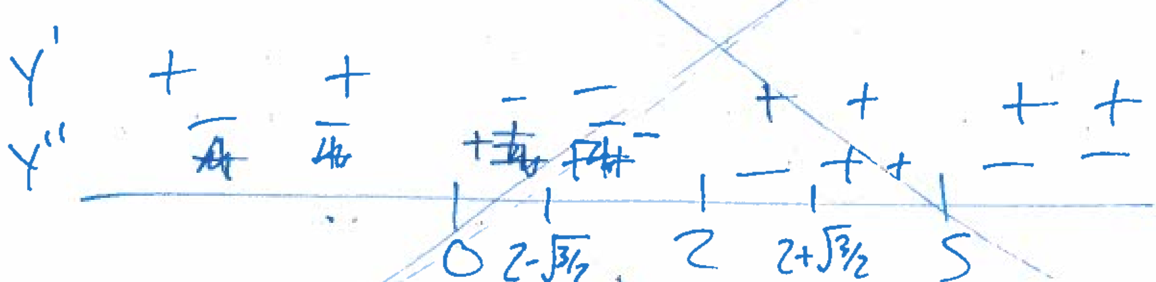


$y' = (x^2-2x)(x-5)^2$ crit. pts at $x = 0, 2, 5$

$y' = (x^2-2x)(x^2-10x+25) = x^4 - 12x^3 + 25x^2$

~~$y'' = (2x-2)(x-5)^2e^x + (x^2-2x)\cdot 2(x-5)\cdot e^x + (x^2-2x)(x-5)^2e^x$~~ $y'' = 4x^3 - 36x^2 + 50x$

~~$= e^x[(x-5)^2(x^2-2) + 2(x^2-2x)(x-5)]$ No inflection pts.~~ $= 4x(x^2 - 9x + 12.5)$



6. **Use L'Hopital's Rule**

- $\lim_{t\to 0} \frac{\sin t^2}{t}$
- $\lim_{x\to 0} \frac{x^2}{\ln \sec x}$
- $\lim_{x\to 1^+} x^{1/(1-x)}$
- $\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x}$
- $\lim_{x\to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$
- $\lim_{x\to 0^+} \csc x - \cot x + \cos x$
- $\lim_{x\to\infty} x^{1/\ln x}$
- $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$
- Find a value c that makes the function

$$f(x) = \begin{cases} \frac{9x-3\sin 3x}{5x^3} & x \neq 0 \\ c & x = 0 \end{cases}$$

continuous at $x = 0$.

$$\lim_{t\to 0} \frac{\sin t^2}{t} \overset{L'H}{=} \lim_{t\to 0} \frac{\cos(t^2)\cdot 2t}{1} = 0$$

$$\lim_{x\to 0} \frac{x^2}{\ln \sec x} \overset{L'H}{=} \lim_{x\to 0} \frac{2x}{\frac{1}{\sec x}\cdot \sec x \tan x} = \lim_{x\to 0} \frac{2x}{\tan x} \overset{L'H}{=} \lim_{x\to 0} \frac{2}{\sec^2 x} = 0$$

$$\lim_{x\to 1^+} x^{1/1-x} \Rightarrow \lim_{x\to 1^+} \ln(x^{1/1-x}) = \lim_{x\to 1^+} \frac{\ln x}{1-x} \overset{L'H}{=} \lim_{x\to 1^+} \frac{1/x}{-1} = -1$$

So $\lim_{x\to 1^+} x^{1/1-x} = \frac{1}{e}$.

$$\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x} \overset{L'H}{=} \lim_{x\to 0} \frac{2x-2}{2x-\cos x} = 2$$

$$\lim_{x\to 0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{(3x+1)\sin x - x}{x\sin x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{(3x+1)\cos x + 3\sin x - 1}{\sin x + x\cos x}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-(3x+1)\sin x + 6\sin x}{\cos x + \cos x - x\sin x} = 0$$

$$\lim_{x\to 0^+} \csc x - \cot x + \cos x = \lim_{x\to 0} \frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x$$

$$= \lim_{x\to 0^+} \frac{1-\cos x + \cos x \sin x}{\sin x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\sin x}{\cos x} + \cos x$$

$$= 1$$

$$\lim_{x\to\infty} x^{1/\ln x} \Rightarrow \lim_{x\to\infty} \ln(x^{1/\ln x}) = \lim_{x\to\infty} \frac{\ln x}{\ln x} \overset{L'H}{=} 1 = \ln(e).$$

So $\lim_{x\to\infty} x^{1/\ln x} = e.$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \Rightarrow \lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\cdot \ln\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}}{-1/x^2} = 1$$

To make f cont. at $x=0$, we must find

$$\lim_{x\to 0} \frac{9x-3\sin(3x)}{5x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{9-9\cos(3x)}{15x^2}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{+27\sin(3x)}{30x} \overset{L'H}{=} \lim_{x\to 0} \frac{81\cos(3x)}{30} = \frac{81}{30} = \frac{27}{10}$$